STRUCTURE OF A SHOCK WAVE IN WHICH MULTIPLE IONIZATION OF THE ATOMS IS TAKING PLACE

N. N. Magretova, N. T. Pashchenko,
and Yu. P. Raizer

Ionization relaxation in a shock wave of very large amplitude is considered, the atoms behind the front of the shock wave being multiply ionized. In calculating the structure of the shock wave and the kinetics of ionization, allowance is made for the electron component of the thermal conductivity which plays an important role in this. A simplified method of calculating the kinetics of multiple ionization is proposed, and an application of this method is presented. The results of the structure calculation show that, as a result of heating by thermal conduction, the gas is considerably ionized even in front of the jump in compression, while the electron component of the thermal conductivity passes through a maximum.

Ionization relaxation behind the leading edge of a shock wave in a gas has been studied both theoretically and experimentally; however, only relatively weak shock waves have been considered, the ionization of the atoms in these being correspondingly weak.* One of the main points of interest is the mechanism of primary ionization, involving the "seed" electrons from which the electron avalanche begins, subsequently developing by way of the ionization of the atoms under the impact of the electrons. In the majority of publications, it has been considered that the primary electrons appear by virtue of atom-atom collisions, although the possibility of initial ionization by virtue of light irradiation has also been considered. An important feature in the process of ionization relaxation is the temperature difference between the electrons and the heavy particles (atoms and ions). In the shock wave, the kinetic energy of the incident flow is converted into thermal energy of the heavy particles; this energy is gradually transferred from the atoms and ions to the electrons, which ionize the atoms. Owing to the retarded exchange of energy between the heavy and light particles, the electron temperature lags behind that of the atoms.

The structure of a shock wave in a partly or completely ionized gas has also been calculated on the assumption of a constant degree of ionization. Such calculations reveal the importance of the electron component of the thermal conductivity, as a result of which the electron gas in front of the jump in compression is heated to a temperature very similar to the equilibrium temperature behind the shock wave. Calculations show that, in severe ionization, the electron thermal conductivity and the exchange of energy between the electrons and ions exert a comparable influence on the structure of the relaxation layer in the shock wave. We note that, in considering the kinetics of ionization, the electron thermal conductivity has never yet been taken into account, possibly because it is not particularly important in weak ionization.

In this paper we shall consider the structure of very strong shock waves with an equilibrium temperature of the order of hundreds of thousands of degrees and over, in which the atoms are multiply ionized. The kinetics of multiple ionization were considered earlier by V. A. Bronshtén and A. N. Chigorin [3] under conditions similar to those existing in shock waves, in connection with the problem of the motion of high-speed meteors through the atmosphere [4, 5]. However, these calculations, although very comprehensive and apparently the only ones of their kind, still fail to provide a complete picture of the real structure

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*An exposition of this problem may be found, for example, in the book of Zel'dovich and Raizer [1]; there is a fairly complete bibliography in a later publication [2].

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of the shock wave, in view of the fact that they omit all consideration of the electron thermal conductivity. As we shall see, this component of thermal conductivity plays a vital role in establishing the electron temperature distribution, which in turn determines the rate of ionization. Hydrodynamic aspects were also omitted from the calculation; the volume of gas was considered constant, and the electron and atomic temperatures were balanced under the assumption of a constant store of total internal energy. (In the subsequent analysis we shall also reformulate the problem of the ionization reaction velocities for ions of different multiplicities and solve it in a novel manner.)

It should be mentioned that the problem here considered differs very greatly in the mathematical respect from the shock-wave structure problems solved earlier. The problem is reduced, in the usual way, to one of finding an integrated curve relating the singular points of the system, the states in the neighborhood of equilibrium and the establishment of equilibrium itself are determined by the behavior of the integrated curve of the system in the neighborhood of the singular points. However, in contrast to the problems treated earlier [6, 7], the kinetics of ionization and the electron thermal conductivity are taken into account at the same time; the result is that the behavior of the curve in the neighborhood of a singular point has to be analyzed, not in a phase plane, but in phase space, and this greatly complicates both the qualitative analysis itself and the numerical calculations. Whereas numerical integration in a plane in the neighborhood of a saddle point presents no serious difficulties, in view of the specific character of the saddle, in the three-dimensional case matters are not so simple: A singular point of the generalized-saddle type constitutes an ordinary saddle in some planes and a node in others, so that a small error in numerical integration may lead to a serious departure from the desired solution.

1. Let us proceed to formulate the problem. Let us consider a plane shock wave in a coordinate system linked to the wave, i.e., the one-dimensional steady motion of the gas in infinite space. In front of the wave the gas is, generally speaking, cold and unionized; behind the wave the atoms are multiply ionized because of the high temperature. The thermodynamical-equilibrium state of the gas behind the wave is completely determined by the initial density and by a certain parameter characterizing the amplitude of the wave, for example, the velocity of the leading edge or the temperature behind it. This state (degree of ionization, density, etc.) may be calculated from the existing relationships governing shock rupture effects and from the thermodynamical functions of the ionized gas [1, 8].*

We shall neglect the viscosity and thermal conductivity of the heavy particles, replacing the viscous layer by a break. The viscosity of the electron gas is also unimportant, but the electron thermal conductivity is much greater than that of the ions and atoms, and acts both in front of and behind the jump in compression. The temperatures of the electron and atomic-ionic gas are regarded as different.

The gas is considered as being monotonic, since in such a strong wave (one involving multiple ionization) the molecules (if the cold gas is molecular) dissociate extremely rapidly and at relatively low temperatures.

Radiation and radiative heat transfer are not taken into account [1]. If the density of the gas is high, let us suppose of the order of the density of atmospheric air, the range of the radiation, determining the spatial scale of the region of radiant heat transfer, is far greater than the zones of ionization relaxation and electron heating in front of the leading edge, which are the only interesting features in the problem under consideration. If, however, the gas is rarefied, the heated region is transparent to the radiation, the density of the radiation within it is far lower than the thermodynamic-equilibrium value, and the radiation has very little effect on the structure of the shock wave.

The question as to the primary ionization of the cold gas does not arise at all in the case of a strong shock wave, since for temperatures of the heated region equal to hundreds of thousands of degrees the radiation is always strong enough to create a small number of electrons in front of the wave, and these initiate the electron avalanche. The structure of the shock wave is almost independent of the extent of the initial ionization.

Let us introduce some notation and definitions. In every state, ions of various multiplicities m are contained in the gas. All the heavy particles will be called ions for brevity, atoms being treated as ions of zero charge $m = 0$.

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*The rupture relationships may be derived from the system of equations describing the structure of the wave by extending the spatial coordinate to infinity in both directions.

Let $N = \Sigma N_m$ be the total number of heavy particles in 1 mm³, where N_m is the density of the m ions; the summation with respect to m always starts from m = 0.

Let $I_1, I_2, \ldots$ be the successive ionization potentials of the atom, $I_0 = 0$, I_{m+1} is the ionization potential of an m ion. The potential energy of the m ion, i.e., the energy expended in its formation from an atom by the detachment of m electrons, is $Q_m = I_1 + I_2 + \ldots + I_m$. The mean potential energy associated with one ion for a given ionic composition

$$\langle Q\rangle = \Sigma\alpha_m Q_m, \quad \alpha_m = \Sigma N_m / N$$

Here α_m is the concentration of m ions. The ions are obtained by electron impacts, and the energy required for this purpose is directly drawn from the store of thermal energy of the electron gas.

Let us call the temperature and pressure of the ions T and p ($p = NkT$), of the electrons T_e and p_e ($p_e = N_e kT_e$), where N_e is the electron density. We regard the gas as electrically neutral, i.e., the electron and ion components are strictly related. Thus, $N_e = \Sigma m N_m$, i.e., the mean charge number of the ions $\langle m\rangle = \Sigma m\alpha_m$ coincides with the degree of ionization of the atoms $\alpha = N_e/N$, while the macroscopic velocities are exactly the same for the two components.

The strictness of the relationship between the electron and ion gas is ensured by the strong polarization Coulomb field E, arising from the slight separation of the charges which in fact always exists. Owing to the negligibly small inertia of the electrons, the Coulomb force acting on the electrons is balanced (to an accuracy limited by small-order electron terms in the ion mass ratio m_e/M) by the force of the electron pressure gradient: $N_e eE = dp_e/dx$. An equal and opposite force acts on 1 cm³ of ion gas.

Subject to the foregoing assumptions, the system of hydrodynamic equations for one-dimensional steady flow in a many-component mixture takes the form [1]:

$$\frac{d}{dx}(N_m u) = q_m \qquad (m = 0, 1, 2\ldots), \qquad \sum_{m=0} N_m = N \tag{1.1}$$

$$MNu\frac{du}{dx} = -\frac{dp}{dx} - \frac{dp_e}{dx} \tag{1.2}$$

$$\frac{d}{dx}\left[N_e u\left(\frac{5}{2}kT_e\right)\right] = -\frac{dS}{dx} + \frac{dp_e}{dx}u + \omega_{ie} - \Sigma Q_m q_m \tag{1.3}$$

$$\frac{d}{dx}\left[Nu\left(\frac{5}{2}kT + \frac{Mu^2}{2}\right)\right] = -\frac{dp_e}{dx} - \omega_{ie}$$
$$\omega_{ie} = \frac{3}{2}\frac{N_e k(T - T_e)}{\tau_{ei}} \tag{1.4}$$

$$S = -\varkappa_e \frac{dT_e}{dx} \tag{1.5}$$

Here q_m is the rate of change of the number of m ions in 1 cm³/sec, associated with the ionization of the m- and (m−1)-ions and the recombination of the m- and (m+1)-ions, S is the flow of heat due to thermal conductivity, $\varkappa_e$ is the electron contribution to the thermal conductivity, the term with the small kinetic energy of the electrons $\frac{1}{2}m_e u^2$ is omitted, ω_{ie} is the rate of energy transfer from the ions to the electrons, and τ_{ei} is the characteristic exchange time.

The system (1.1)-(1.5) has three integrals expressing the laws of conservation of the mass, momentum, and energy flows. The first integral is obtained if we sum all Eqs. (1.1) and consider that the reaction does not alter the total number of particles

$$Nu = N_0 D \tag{1.6}$$

where N_0 is the number of ions in 1 cm³ of the original gas (the index 0 refers to quantities in front of the wave), D is the velocity of propagation of the shock wave through the cold gas. The second integral following from (1.1) and (1.6):

$$MNu^2 + p + p_e = MN_0 D^2 + p_0 + p_{e0} \tag{1.7}$$

In order to obtain the third integral we add Eqs. (1.3) and (1.4) and use (1.1) and (1.6)

$$\frac{5}{2}(kT + \alpha kT_e) + \langle Q\rangle + \frac{Mu^2}{2} + \frac{S}{N_0 D} = \frac{5}{2}(kT_0 + \alpha_0 kT_{e0}) + \langle Q_0\rangle + \frac{MD^2}{2} \tag{1.8}$$

Equation (1.6) enables us to rewrite the continuity equation in the form of kinetic equations for the concentrations:

$$N_0 D \frac{d\alpha_m}{dx} = q_m \qquad (m = 0, 1, 2, \ldots) \tag{1.9}$$

The system of differential equations representing the ion energy (1.1), the thermal-conductivity heat flow (1.5), the ionization kinetics (1.9), and the conservation integrals (1.6)-(1.8), together with the boundary conditions which we shall be discussing later, serves to determine the structure of the shock wave.

2. Let us simplify the system of equations and set out the kinetic constants. A knowledge of the ionic composition of the gas α_m is necessary in order to find the degree of ionization $\alpha = \Sigma \alpha_m m$ and the potential energy of the ions $\langle Q \rangle$. However, the kinetic equation may also be written directly for the degree of ionization on the basis of (1.9)

$$N_0 D \frac{d\alpha}{dx} = q_e = \Sigma m q_m \tag{2.1}$$

Hence if we could approximately express the electron source q_e and the mean potential energy of the ions $\langle Q \rangle$ as functions, not of the set of ion concentrations α_m, but simply of the degree of ionization α, this would enable us to replace the set of equations (1.9) by the single equation (2.1), and this would considerably simplify the original cumbersome set.

We may carry out this operation by the method earlier proposed by one of the authors [1,9] for simplifying the calculation of thermodynamic functions of gases in the region of multiple ionization. Let us suppose that the charge number m, which actually only assumes integral values, instead constitutes a continuously varying quantity, and let us join the discrete I_m and Q_m by continuous curves $I(\alpha)$ and $Q(\alpha)$. Usually ionization develops by the successive uncovering of the atoms, so that at any moment a gas particle contains appreciable numbers of ions of only two or at most three types. In other words, the distribution α_m, which is replaced by the continuous function, takes the form of a peak in m relative to the mean value $\langle m \rangle$. We may thus approximately consider that in each state all the ions possess a single fractional charge $\langle m \rangle$, coinciding with the degree of ionization α. In this approximation the ionization potential of equivalent ions may be regarded as $I(\alpha)$, while the mean potential energy $\langle Q \rangle = Q(\alpha)$. In the region of primary ionization, for $\alpha < 1$ it is natural to put $I = I_1$, $Q = \alpha I_1$.

The electron source q_e constitutes the difference between the rates of ionization and recombination of the ions. There may be two ways of ionizing atoms and ions by an electron beam: 1) the direct removal of electrons from the ground level and 2) ionization in a number of steps, in which the atom or ion is first excited by electron impact and then ionized either immediately or after several more exciting collisions. Estimates show that the second mode is frequently the more effective, since the excitation and ionization cross sections increase rapidly on raising the level. We shall not consider ionization from the ground level, electron capture at the latter, or photo-recombination (although the latter may occur when the density is not extremely low).

Estimates show that equilibrium between the population of the excited states of the ions (atoms) and the free electrons is established far more quickly than the electron concentration changes. This means that the distribution of the ions over the excited levels may be regarded as approximately quasi-stationary, assuming that it follows a comparatively slow development of ionization. Under such quasi-stationary conditions, which are satisfied if the ionization potential of the ions is much greater than kT_e and the number of excited particles is far smaller than the number of unexcited particles, the appearance or disappearance of every electron is accompanied by the vanishing or formation of a corresponding ion in the ground state, without any time lag or lead.

However, the immediate cause of the vanishing and formation of ions in the ground state lies in their excitation and deactivation by electron impacts,* so that

$$q_e = K^* N_e N - K' N_e N^* = K^* N_e N (1 - N^* / N_p^*)$$

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*We shall not consider deactivation by scintillation; this is compensated by excitation after the absorption of captured resonance radiation.

where K* and K' are the velocity constants of excitation and deactivation, N* is the density of the excited ions, N^*_p is a thermodynamically equilibrium quantity; the foregoing equation follows from the principle of detailed equilibrium. Since N* is in thermodynamic equilibrium with the actual density of the electrons, under conditions of multiple ionization $N^* \sim N_e N$, so that finally

$$N_0 D \frac{d\alpha}{dx} = K^* \alpha N^2 \left(1 - \frac{\alpha}{\alpha_p}\right) \tag{2.2}$$

where α_p is the thermodynamic-equilibrium degree of ionization. On the approximation of "equivalent" ions, with a fractional charge, this quantity [1, 9] is determined from the equation

$$I(\alpha) = kT_e \ln \frac{AT_e^{3/2}}{\alpha N}, \qquad A = 2\left(\frac{2\pi m_e k}{h^2}\right)^{3/2} \tag{2.3}$$

Equation (2.2) may be written down directly from the expression for the difference between the ionization and recombination rates, the principle of detailed equilibrium, and the assumption that the rates of ionization and excitation coincide (the latter corresponds to the instantaneous ionization of excited atoms). This derivation gives a truer picture of the physical meaning of Eq. (2.2).*

The excitation velocity constant equals $K^* = \langle v_e \sigma(\varepsilon)\rangle$, where $\sigma(\varepsilon)$ is the excitation cross section for electrons with energy ε, v_e is their velocity, the angular brackets denote averaging over the Maxwell distribution. In the case of ions, the cross section is almost constant at the threshold $\sigma(\varepsilon) = \sigma(I^*) = \sigma$ (I* is the excitation potential), and

$$K^* = \sigma \langle v_e \rangle \left(\frac{I^*}{kT_e} + 1\right) \exp\frac{-I^*}{kT_e}, \qquad \langle v_e \rangle = \left(\frac{8kT_e}{\pi m_e}\right)^{1/2} \tag{2.4}$$

where $\langle v_e \rangle$ is the mean thermal velocity. In neutral atoms the cross section increases linearly from the threshold, $\sigma = \sigma(\varepsilon) = C(\varepsilon - I^*)$, and instead of (2.4) we obtain a scarcely different formula in which the 1 in the brackets is replaced by 2 while $\sigma = \sigma(kT) = CkT$.

Published data regarding the excitation cross sections of ions are so sparse and imperfect that we can here only make a very rough choice of σ.

According to the results of calculations based on the Born approximation for a Coulomb field, carried out in relation to hydrogen-like ions [10], we may put $\sigma = 1.8\pi a_0^2/(m + 1)^4$, where a_0 is the Bohr radius and and $m \geq 1$.

The quantity σ is rather too high; in [10] it is recommended that its value should be reduced by a factor of 5. Furthermore, if we consider the origin of the charge factor $(m + 1)^{-4}$, in applying the equation to nonhydrogen-like ions it is natural to replace the $(m + 1)^{-4}$ by the square of the ratio of the "hydrogen-like" ionization potential to the true one; $(m + 1)^{-4} \rightarrow (I_H/I)^2$. As excitation potential we shall take the value $I^* = {}^3/_4 I$, as in the case of hydrogen-like atoms. Neglecting unity in comparison with I^*/kT_e, we obtain

$$K^* = 0.54\sqrt{2\pi}\, a_0^2 \left(\frac{I_H^2}{I}\right)\left(\frac{1}{m_e kT_e}\right)^{1/2} \exp\frac{-3I^*}{4kT_e} \tag{2.5}$$

This equation is, generally speaking, unsuitable for describing the excitation of atoms [1]; however, it gives acceptable results if we take the true value of I_1 for the potential (for $\alpha < 1$). This appears reasonable in view of the fact that a variation in the pre-exponential factor in the equation for K* has very little effect on the calculated structure of the shock wave, particularly in the less important region of primary excitation, as special calculations confirm.

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*In the region of weak, single ionization, the equation corresponding to (2.2) has a slightly different form, since the rate of recombination is proportional to $N_e N_+ N_e \sim \alpha^3$ and not α^2 as in (2.2) ($N^* \sim N_e N_+ \sim N_e^2$ and not $N_e N$). The term in brackets equals $(1 - \alpha^2/\alpha_p^2)$ instead of $(1 - \alpha/\alpha_p)$. However, recombination only plays an appreciable part in the shock wave on approaching the final state of the multiply ionized gas; hence Eq. (2.2) may be everywhere used in the wave.

As regards the remaining two kinetic constants, the time required for energy exchange between the electrons and ions τ_{ei} and the electron thermal conductivity $\varkappa_e$, these were calculated from existing relationships [1]:

$$\tau_{ei} = \frac{3\sqrt{m_e}\,(kT_e)^{3/2}}{4\sqrt{2\pi}\,\Lambda e^4 z N_e}, \quad \varkappa_e = \zeta \frac{k\,(kT_e)^{5/2}}{\sqrt{2\pi}\,\Lambda e^4 \sqrt{m_e}\,z} \tag{2.6}$$

where Λ is the Coulomb logarithm, which is here regarded as constant and equal to its value in the final state, z is the charge number of the ions, the coefficient ζ depends very slightly on z and is taken as equal to $\zeta = 5.25$.

The specification of the initial state of the gas in front of the wave and of some parameter characterizing the amplitude of the shock wave, for example, the wave propagation velocity D, makes the mathematical problem of solving the system completely definite. Since the equation does not include any terms describing the sources of the original electrons, we must specify a small but finite initial degree of ionization in front of the wave. For the same reason the initial temperatures and pressures $T_0 = T_{e0}$ and p_0, $p_{e0} = \alpha p_0$ must be regarded as nonzero, although in the very strong waves considered they are far smaller than the corresponding quantities behind the leading edge. The specified initial values of T_{e0}, α_0, and N_0 are not independent but have to satisfy the Saha equation in order to make the initial state of the gas thermodynamically equilibrium. The final values of the parameters T_k, N_k, α_k are determined in the usual way from the initial values and the wave velocity D.

3. Let us examine the resultant system of equations qualitatively. We convert to dimensionless variables

$$\xi = \frac{x}{L_i}, \quad \theta_e = \frac{kT_e}{2\varepsilon_0}, \quad \theta = \frac{kT}{2\varepsilon_0}, \quad i = \frac{I}{2\varepsilon_0}, \quad q = \frac{Q}{2\varepsilon_0}, \quad n = \frac{N}{N_0}$$

taking as scale of particle energy the initial kinetic energy of the atom $\varepsilon_0 = MD^2/2$ and as scale of length the characteristic ionization length corresponding to the final state behind the leading edge of the wave $L_i = N_0 D/K^* N_k^2$ as in Eq. (2.2).

Instead of the pressures p and p_e we substitute their values $p = NkT$ and $p_e = N_e kT_e$ and eliminate u by means of (1.6). Solving the system for the derivatives of temperature and degree of ionization and making a number of further transformations, we finally obtain a system of three differential equations of the first order

$$\begin{aligned}
\frac{d\alpha}{d\xi} &= f_1(\alpha, \theta, \theta_e) = \frac{i_k}{i}\left(\frac{\theta_k}{\theta_e}\right)^{1/2}\left(\frac{n}{\delta}\right)^2 \alpha \exp\left[-\frac{3}{4}\left(\frac{i}{\theta_e} - \frac{i_k}{\theta_k}\right)\right]\left(1 - \frac{\alpha}{\alpha_p}\right) \\
\frac{d\theta_e}{d\xi} &= f_2(\alpha, \theta, \theta_e) = \frac{2}{5}\frac{L_i}{L_s}\frac{z}{\alpha_k}\left(\frac{\theta_k}{\theta}\right)^{5/2}\left[\frac{5}{2}(\vartheta - \vartheta_0) + \frac{1}{2}\left(\frac{1}{n^2} - 1\right) + q - q_0\right] \\
\frac{d\theta}{d\xi} &= f_3(\alpha, \theta, \theta_e) = F\frac{L_i}{L_\omega}\frac{\alpha^2 z}{\alpha_k^3}\left(\frac{n}{\delta}\right)^2\left(\frac{\theta_k}{\theta_e}\right)^{3/2}(\theta_e - \theta) + \frac{2Fn^2\theta}{2 - an}(\alpha f_2 + \theta_e f_1)
\end{aligned} \tag{3.1}$$

and an algebraic equation for the density

$$n^2\vartheta - an + 1 = 0 \tag{3.2}$$

Here we have introduced the following notation:

$$\vartheta = \theta + \alpha\theta_e, \quad a = 1 + \vartheta_0, \quad \delta = \frac{N_k}{N_0}, \quad F = \left(3 - \frac{2n^2\theta}{2 - an}\right)^{-1} \tag{3.3}$$

The characteristic scales of length

$$L_s = \frac{2}{5}\frac{\varkappa_e}{\alpha_k k N_0 D}, \quad L_\omega = \frac{2}{3}\frac{D\tau_{ei}}{\alpha_k \delta}$$

for the thermal conductivity and the exchange of energy between the electrons and ions correspond to the final state of the gas.

Choosing as independent variable the monotonically varying quantity α, we reduce the system (3. 1) to two equations in phase space

$$\frac{d\theta_e}{d\alpha} = \frac{f_2(\alpha, \theta, \theta_e)}{f_1(\alpha, \theta, \theta_e)}, \qquad \frac{d\theta}{d\alpha} = \frac{f_3(\alpha, \theta, \theta_e)}{f_1(\alpha, \theta, \theta_e)} \tag{3.4}$$

the density n being related to the phase variables by the same final equation (3.2).

Thus, in order to solve the problem of the structure of the shock wave, we must find the integrated curve of the system (3.4) in phase space α, θ, θ_e connecting the equilibrium points α_0, θ_0, θ_{0e} and α_k, θ_k, θ_{ek} in front of and behind the wave. These points are singular points of the system, since all the f_i vanish simultaneously in them (it may be shown that there are no other singular points in the system). In order to find the curve in question, we must know the behavior in the neighborhood of the singular points, since this describes the near-equilibrium states and the process of equilibrium establishment.

The behavior of the integrated curve in the neighborhood of a singular point is determined by the character of the singularity; the ascription of a singular point to one particular type of character, in turn, is determined by the roots of the characteristic equation $|a_j^i - \lambda_j^i| = 0$, where a_j^i are the derivatives of the right-hand sides of the system of f_i with respect to their arguments y^i calculated at the singular points. In the case under consideration the arguments are α, θ, θ_e. Here δ_j^i is the Kronecker delta.

The characteristic equation of the system under consideration is a third-order equation

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$$

where J_k are invariants of the coefficients of the matrix $\|a_j^i\|$.

It is easy to see [from Eqs. (3.1), (3.2)] that the coefficients a_j^i (and hence J_k) are expressed in terms of the equilibrium parameters and the chosen characteristic lengths. Corresponding estimates show that, in the near-equilibrium state, in front of the wave the most rapid process (the shortest characteristic length) is the heating of the electron gas by thermal conduction, and the slowest is the exchange of energy between the electrons and the heavy particles. These estimates indicate the considerable effect of the electron thermal conductivity on the structure of the wave; the heating of the electron gas in front of the wave may lead to considerable ionization in this region. In the near-equilibrium state, the most rapid process behind the wave is energy exchange (which tends to equate the electron and ion temperatures), while the slowest is ionization.

Let us return to an analysis of the roots of the characteristic equation. The type of singular point is determined by the signs of the real parts of these roots, and the signs may be determined by using, for example, the Rauss theorem, according to which the number of roots of a real polynomial lying in the right-hand half plane is equal to the number of changes in sign in the first column of the Rauss scheme set up in a particular fashion [11].

In the case under consideration this column takes the form

$$1, \quad -J_1, \quad J_2 - J_3 / J_1, \quad -J_3 \tag{3.5}$$

Thus we have to determine the signs and orders of magnitude of the coefficients of the characteristic equation; this may be done by using the resultant relationships between the characteristic lengths and orders of magnitude of the equilibrium parameters. The calculations are very cumbersome and will not be presented in detail: the results show that for both the singular points the sign in the Rauss column (3.5) only changes once, i.e., the signs of the real parts are different (one root in the right- and two in the left-hand half plane), and the two singular points constitute generalized saddles [12]. A more detailed investigation shows that these generalized saddle points constitute three-branch saddle-nodes with stable two-branch nodes [13].

It is well known that, in shock-wave-structure problems in which the equation makes no allowances for viscosity, a continuous curve joining the singular points (which in the present case are saddle points) does not in general exist. There is a break within the structure, a sharp jump in compression at which, in the present case, the ion temperature, the density, and hence the velocity undergo a discontinuity, while the electron temperature and the degree of ionization are continuous. The position of the discontinuity in phase space is determined by the condition that, on passing through the jump, the quantities undergoing the discontinuity should satisfy an equation of the Rankine–Hugoniot type.

As already indicated, in the characteristic equation there is one root in the right-hand half plane and two in the left. This means that one integrated curve passes out from a point in front of the wave while two pass to a point behind the wave; in order to choose the one corresponding to the problem in hand, we have to study the asymptotic of the solution in the neighborhood of this point. Using the foregoing relations between the characteristic lengths, expressing the fact that in the near-equilibrium state exchange takes place "faster" behind the wave than ionization-recombination or heat transfer by thermal conduction ($L_\omega \ll L_s \ll L_i$), so that in this region the temperatures are practically identical, we may write the asymptotic of the system in the form

$$\theta = \theta_e$$

$$\frac{d\theta}{d\alpha} = -\frac{\theta}{1+\alpha}\left\{1 + \frac{2i\,(2-an)}{\theta\,[5\,(2-an)-2]}\right\}, \qquad n = \frac{a\,(1+\sqrt{1-4\vartheta/a^2})}{2\vartheta} \tag{3.6}$$

Clearly both temperatures, while remaining identical, will fall to their equilibrium value ($d\theta/d\alpha < 0$), and hence the electron temperature in the wave will pass through a maximum (in the earlier analyses it was asserted that the electron temperature increased monotonically in the wave).

Thus, of the two integrated curves proceding to the point behind the wave, we must choose the one for which the temperatures of both components are close together and fall on approaching equilibrium.

The foregoing qualitative analysis enables us to integrate the system of equations describing the structure of the shock wave numerically. Whereas, in a phase plane, the saddle is an ideal form for starting the calculation, in phase space the matter is more complicated, since the generalized saddle constitutes an ordinary saddle in some planes and a node in others, so that a slight deviation in the calculation may carry us away from the solution desired. The computing process was as follows. The two parts of the integrated curve were calculated starting from the singular points and moving toward each other until they met in the "viscous" jump. The conservation conditions at the viscous jump may be transformed to relationships linking the dimensionless ion densities and temperatures in front of and behind the jump (the parameters without any index characterize the state of the gas in front of the jump; behind the jump they bear the index 1):

$$n_1 = \frac{4n}{5n^2\vartheta + 1}, \qquad \theta_1 = \theta + \frac{1}{5}\left(\frac{1}{n_1^2} - \frac{1}{n^2_1}\right) \tag{3.7}$$

For each point of the integrated curve in front of the jump α, θ, θ_e and hence for each value of n determined by Eq. (3.2), the values of n_1 and θ_1 were determined from (3.7). The set of all these values forms the geometrical locus of the points in phase space satisfying the Rankine–Hugoniot equations.

The intersection of the integrated curve starting from a point behind the jump with the resultant geometrical locus gives the position of the jump in phase space and the true parameters behind the jump n_1, θ_1, α, θ_e and hence also their values in front of the jump n, θ, α, θ_e. In calculating the integrated curve behind the jump, it is more convenient to start from the final singular point by the asymptotic system (3.6),

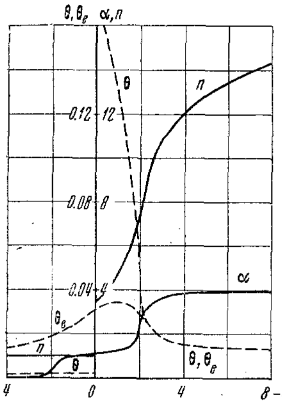

Fig. 1

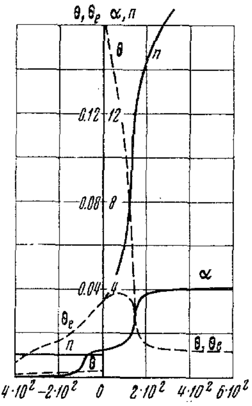

Fig. 2

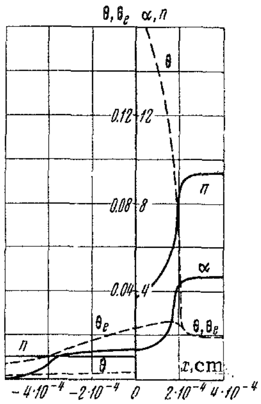

Fig. 3

for which this point is not singular. In calculating the density by the quadratic equation (3.2) in the region behind the jump, the radical must be taken with plus sign, and in front of the jump with the minus sign. The distribution of the parameters with respect to the coordinate ξ or x may be obtained after solving the equations in phase space by quadrature.

4. Let us now discuss the results of the calculations. The structure of the shock wave was calculated for air, using several values of the initial density and velocity of the shock wave. The curves of the ionization potentials $I(\alpha)$ and the potential energies of the ions $Q(\alpha)$ were obtained by means of Lagrange interpolation polynomials from the discrete values corresponding to the integral values of z, these being determined from tables [8]. The initial degree of ionization was taken as $\alpha_0 = 0.01$ in all the calculations. Special checks showed that a variation in α_0 (within reasonable limits) had very little effect on the solution.

Figures 1-3 illustrate the temperature, density, and degree-of-ionization distributions for the three following versions of the initial data:

h, km	N_0, cm^{-3}	D, km/sec	T°, K
80	$3.5 \cdot 10^{15}$	57.9	9 000
100	$4.2 \cdot 10^{13}$	56	7 500
0	$5.4 \cdot 10^{20}$	75	16 783

Here h is the height above sea level corresponding to the initial air density N_0. The resultant distributions confirm the substantial effect of the electron thermal conductivity on the structure of the wave; the heating of the electron gas in front of the wave is so considerable that the first ionization takes place even before the jump. In this respect there is a considerable difference between this and the earlier analyses [3, 5], in which, in calculating the relaxation zone, the authors started from the values of the ion temperature behind the viscous jump, and in calculating the electron concentration and temperature from their unperturbed values: Initial computing data of this kind could hardly fail to affect the distribution of the parameters in the wave.

Yet another special feature is the maximum in the electron-temperature profile. The existence of this maximum is due to the rapid exchange taking place in this region, which leads to the balancing of the temperatures even before the completion of ionization, while cooling continues as a result of the consumption of energy in the ionization process. The graphs indicate quite clearly that the balancing of the temperatures takes place immediately after the maximum point on the descending branch of the electron-temperature profile, this being a region of very high gradients in all the parameters. The density of ions and the electron concentration increase very sharply in this region, while the gas temperature falls sharply, the thermal-conduction flow being directed to the rear. Beyond the region of high gradients there follows a slow approach of the parameters to their equilibrium values, and in this region the contribution of the recombination process becomes considerable.

The calculations were carried out in the BÉSM-3M computer.

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